Motivation

$$-\Delta u = f \quad \text{on } B \qquad\qquad u = 0 \quad \text{on } \partial B \longrightarrow \begin{cases} B(u,\phi) = \int_B Du\, D\phi = \int_B f\phi = \langle f,\phi\rangle \\ u=0 \text{ on } \partial B \end{cases}$$

Weak form, no longer require $u \in C^2$.

- We may define a norm $\langle u,v\rangle_{H_0^1} = \langle Du, Dv\rangle_{L^2}$

  linear functional $L: C_c^\infty(B) \to \mathbb{R}$

  $L(\phi) = \int_B f\phi\, dx$

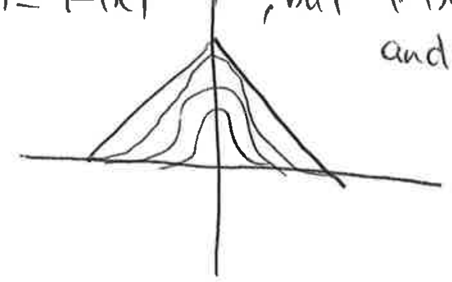
- ~~L can be shown~~ L is bdd w.r.t. $\cancel{\|\cdot\|_{H_0^1}}$ H_0^1-norm. (Poincare).

Thus Weak form $\Leftrightarrow$ find $u \in C^\infty(\overline{B})$ with $u=0$ on ∂B s.t.

$$\begin{cases} \langle u,\phi\rangle_{H_0^1} = L(\phi) \quad \forall \phi \in C_c^\infty(B) \\ L \text{ bdd w.r.t } H_0^1\text{-norm.} \end{cases}$$

- Well, apply RRT, but no, $(C_c^\infty(B), \|\cdot\|_{H_0^1})$ is not complete!

  Indeed, can find $u_m \in C_c^\infty([-1,1])$ s.t. $u_m \to u_\infty$,

  where $u_\infty(x) = 1-|x|$ , but $1-|x|$ is not smooth and not compactly supported.

To fix this, we take the completion of C_c^∞ w.r.t. $\|\cdot\|_{H_0^1}$

$X = \{\text{Cauchy seq. in } C_c^\infty\}$ w.r.t. $\|\cdot\|_{H_0^1}$

equivalence relation $\sim$ on X as

$\{u_i\} \sim \{v_i\} \Leftrightarrow \text{dist}(u_i, v_i) \to 0$ as $i \to \infty$

Weak Derivatives

Given $u \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, IBP shows that

$$\int_\Omega u\, \partial_{x_i}\phi\, dx = \int_\Omega \partial_{x_i} u\, \phi\, dx \quad, i=1,\dots,n \quad \forall \phi \in C_c^\infty(\Omega).$$

The boundary terms vanishes since ϕ has compact support in Ω and so it vanishes near $\partial\Omega$. More generally, if $u \in C^k(\Omega)$, then for $|\alpha| \le k$,

$$\int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u\, \phi\, dx \quad \forall \phi \in C_c^\infty(\Omega)$$

this makes sense if $u \in L^1_{loc}(\Omega)$

This is unclear if $u \notin C^k(\Omega)$......

$C^k(\Omega) = \{u : \Omega \to \mathbb{R} : D^j u, j=0,\dots,k$ exists and are continuous in $\Omega\}$

$C_c^\infty(\Omega) = \{u \in C^\infty(\Omega)$ with compact support in $\Omega\}$.

Compact support means $\{x \in \Omega : u(x) \neq 0\}$ is a compact subset of Ω.

Definition: If $u \in L^1_{loc}(\Omega)$ and α is a multiindex, we say that $v \in L^1_{loc}(\Omega)$ is the αth weak derivative of u, written $v = D^\alpha u$, if

$$\int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega v\, \phi\, dx \quad \forall \phi \in C_c^\infty(\Omega).$$

Lemma: (Uniqueness) ~~With u, α as above, if $\exists v_1, v_2 \in L^1_{loc}(\Omega)$ s.t. αth weak derivative~~ With u, α as above, if $v_1, v_2 \in L^1_{loc}(\Omega)$ are both αth-weak ~~deriv~~ derivatives of u, then $v_1 = v_2$ a.e., i.e. $v_1 = v_2$ up to a set of measure zero.

Proof: We have that $$\int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega v_1 \phi\, dx = (-1)^{|\alpha|} \int_\Omega v_2 \phi\, dx \quad \forall \phi \in C_c^\infty(\Omega)$$

$$\Rightarrow \int_\Omega (v_1 - v_2)\phi\, dx = 0 \quad \forall \phi \in C_c^\infty(\Omega)$$

$$\Rightarrow v_1 - v_2 = 0 \text{ a.e.} \quad \text{(Fundamental lemma of CoV).}$$

$\boxed{/\!/}$

$L^1_{loc}(\Omega) = \{u : \Omega \to \mathbb{R}$ measurable $: u|_K \in L_1(K)\ \forall K \subset\subset \Omega\}$

compactly contained, i.e. $K \subset \Omega$ and K is compact

Remarks: (1) It should be clear that if $u\in C^k(\Omega)$, then for $|\alpha|\le k$, the weak derivative $D^\alpha u$ exists and coincides with the classical derivative.

(2) We can go stronger:

for $\tilde\Omega\subset\Omega\subset\mathbb{R}^n$ open, if $D^\alpha u$ exists on Ω, then $D^\alpha(u|_{\tilde\Omega})$ exists and $D^\alpha(u|_{\tilde\Omega})=(D^\alpha u)|_{\tilde\Omega}$. In particular, if $u\in C^k(\tilde\Omega)$ and $D^\alpha u$, $|\alpha|\le k$, exists, then $D^\alpha u$ must agree with the classical derivative on $\tilde\Omega$.

$D^\alpha u$ exists $\Rightarrow$ $(D^\alpha u)|_{\tilde\Omega}$ = classical derivative of u on $\tilde\Omega$, $|\alpha|\le k$.

Example: Let $\Omega=(0,2)\subset\mathbb{R}$ and define $u(x)=\begin{cases} x & \text{if } x\in(0,1]\\ 1 & \text{if } x\in[1,2).\end{cases}$

Weakly differentiable but not classically differentiable

We claim that weak derivative $u'=v$ exists and is $v(x)=\begin{cases}1 & \text{if } x\in(0,1]\\ 0 & \text{if } x\in(1,2).\end{cases}$

Proof: Choose $\phi\in C_c^\infty(\Omega)$, NTS $\int_0^2 u\phi'\,dx=-\int_0^2 v\phi\,dx$.

$$\int_0^2 u\phi'\,dx=\int_0^1 x\phi'\,dx+\int_1^2\phi'\,dx$$
$$=x\phi\Big|_0^1-\int_0^1\phi'\,dx+\cancel{\phi(2)}-\phi(1)$$
$$=-\int_0^1\phi'\,dx+\phi(1)-\phi(1)=-\int_0^1\phi\,dx=-\int_0^2 v\phi\,dx.$$
$\square$

Example: It seems like it is ~~Ok~~ enough to be classically diff except at 1 point, but this is not enough!

This is example 2, pg 257, but we give a simpler example. Define.

$\cancel{u(x)=\begin{cases}1 & \text{if } x\in(0,1]\\ 0 & \text{if } x\in\end{cases}}$ $\quad u(x)=\begin{cases}0 & \text{if } x\in(-1,0)\\ 1 & \text{if } x\in(0,1).\end{cases}$, $\Omega=(-1,1)$.

$$-\int_{-1}^{1} v\phi\, dx = \int_{-1}^{1} u\phi' dx = \int_0^1 \phi' dx = \cancel{\phi(1)} - \phi(0) = -\phi(0) \quad \forall \phi \in C_c^\infty(\Omega)$$

$$\Rightarrow \int_{-1}^{1} v\phi\, dx = \phi(0) \quad \forall \phi \in C_c^\infty(\Omega). \quad \text{---} (\#)$$

Taking $\phi \in C_c^\infty(\Omega)$ with $\phi(0) = 0$ implies that $v \equiv 0$ a.e., but then $(\#)$ fails to hold for $\phi \in C_c^\infty(\Omega)$ with $\phi(0) \neq 0$. $\blacksquare$

Intuitively $u' = v$ wants to be Dirac delta, which is a distributional derivative; δ cannot be "represented" by L^1_{loc} functions

In fact, for $n = 1$, if u is classically diff except at 1 point, then

u is weakly diff $\Leftrightarrow$ u is continuous at that point

Example: (Cts $\not\Rightarrow$ weakly differentiable)

Consider the Cantor's function $u: [0,1] \to \mathbb{R}$ which is continuous, non-decreasing, but not absolutely continuous, with $u(0) = 0$, $u(1) = 1$. It is classically differentiable, with derivative equal to 0, on an open subset of $[0,1]$ of full measure ~~[Complement~~, specifically, the complement of Cantor set in $[0,1]$: $(\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}), \ldots = [0,1] \setminus C$

Suppose $u' = v$, where $\int_0^1 u\phi' dx = -\int_0^1 v\phi\, dx \quad \forall \phi \in C_c^\infty [0,1]$.

Choose $\phi \in C_c^\infty$ whose support is in one of these intervals, call it I. Since u is constant on I, $-\int_0^1 v\phi\, dx = \int_0^1 u\phi' dx = C \int_I \phi' dx = C\phi\big|_I = 0.$

$\Rightarrow$ $v = 0$ pointwise a.e. on $[0,1] \setminus C$.

$\Rightarrow$ $v = 0$, if u were to be weakly differentiable.

This is a contradiction, since the only functions with zero weak derivatives are constant functions (a.e.). $\blacksquare$

Thm: Let $\Omega \subset \mathbb{R}^n$ be open and connected. If $u \in L^1_{loc}(\Omega)$ is weakly differentiable and $Du = 0$ a.e., then $u =$ constant a.e.

** It is important to include weakly differentiable in the defn of "weak solution". O/w, ~~step~~ Heaviside function and Cantor function satisfy $u' = 0$ a.e., but they are not weakly differentiable !!

Example: (Weakly diff $\not\Rightarrow$ continuous).

open unit ball in $\mathbb{R}^n$.

Consider the ~~open unit~~ function $u : B_1(0) \to \mathbb{R}$, $u(x) = |x|^{-\alpha}$, $\alpha > 0$, $x \neq 0$.

We claim that it is weakly differentiable if $\alpha \in (0, n-1)$.

Away from the origin, u is smooth and $D_i u = \dfrac{-\alpha x_i}{|x|^{\alpha+2}}$, $x \neq 0$, $i = 1, \dots, n$.

So if u is weakly diff, then this would be its weak derivative. We just need to prove that this is locally integrable.

Let us ~~look at~~ cut an ε-ball around 0 and look what happens [Common trick].

Using divergence thm, $\forall \phi \in C_c^\infty(B_1(0))$, we have that

$$\int_{B \setminus B_\varepsilon(0)} u D_i \phi \, dx = -\int_{B \setminus B_\varepsilon(0)} D_i u \phi \, dx + \int_{\partial B_\varepsilon(0)} u \phi \nu^i \, dS,$$

$\nu = (\nu^1, \dots, \nu^n)$ ~~$\nu = (\nu_1, \dots, \nu_n)$~~ denoting the inward pointing normal on $\partial B_\varepsilon(0)$.

We estimate the boundary term:

$$\left| \int_{\partial B_\varepsilon(0)} u \phi \nu^i \, dS \right| \leq \|\phi\|_{L^\infty} \int_{\partial B_\varepsilon(0)} \varepsilon^{-\alpha} \, dS$$

$$\leq \|\phi\|_{L^\infty} \operatorname{vol}(\partial B_\varepsilon(0)) \, \varepsilon^{-\alpha}$$

$$\lesssim C \varepsilon^{n-1-\alpha} \to 0 \text{ as } \varepsilon \to 0 \text{ if } n-1-\alpha > 0$$

C only depends on the dimension. $\qquad n-1 > \alpha$.

Sending $\varepsilon \to 0$ then yields $\displaystyle\int_B u D_i \phi \, dx = -\int_B (D_i u) \phi \, dx$ $\forall \phi \in C_c^\infty(B)$, provided $\alpha \in (0, n-1)$.

$\boxed{//}$

It might appear that weakly diff functions are smooth except on some "small" set.
But no!! They are truly ~~weird~~ weird ......

Example: (Weakly diff ~~⇏~~, ~~b~~ but unbounded on any open set).

Let $\{r_k\}_{k=1}^{\infty}$ be a countable, dense subset of $\Omega = B_1(0)$.

Write
$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\alpha}, \quad x \in B_1(0)$$

Despite being regular enough to be weakly differentiable

For $\alpha \in (0, n-1)$, one can show that $u(x)$ is weakly differentiable and yet it is unbounded on any open subset of $B_1(0)$, however small.

Lemma: Suppose $u : (-1,1) \to \mathbb{R}$ and it is classically differentiable except at 1 point. Then u is weakly diff $\Longleftrightarrow$ u is continuous at that point.

Proof: WLOG, suppose $u_1 = u|_{(-1,0)}$ and $u_2 = u|_{(0,1)}$ are differentiable, with derivatives u_1', u_2'.

If u is weakly differentiable, then u' should be $u_1' + u_2'$.

$$\begin{aligned}
\int_{-1}^{1} (u_1' + u_2')\phi \, dx &= \int_{-1}^{0} u_1'\phi \, dx + \int_{0}^{1} u_2'\phi \, dx \\
&= u_1\phi \Big|_{-1}^{0} - \int_{-1}^{0} u_1\phi' dx + u_2\phi \Big|_{0}^{1} - \int_{1}^{0} u_2\phi' dx \\
&= \int_{-1}^{1} (u_1 + u_2)\phi' dx + \left[u_1(0)\phi(0) - u_2(0)\phi(0) \right] \\
&= \int_{-1}^{1} (u_1 + u_2)\phi' dx \implies u_1(0) = u_2(0)
\end{aligned}$$

$|\alpha| \int |x|^{-(\alpha+1)p}$

$= r^{-(\alpha+1)p} r^{n-1}$

$= r^{n-(\alpha+1)p-1} > -1$

$n - (\alpha+1)p > 0$

$(\alpha+1)p < n$

Sobolev spaces

Definition: For $\Omega \subset \mathbb{R}^n$ open, $p \in [1,\infty]$, $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$, the Sobolev space $W^{k,p}(\Omega)$ is the vector spaces of all functions $u \in L^p(\Omega)$ s.t. for all multiindices α with $|\alpha| \le k$, the weak derivatives $D^\alpha u$ exists and is also in $L^p(\Omega)$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \text{ exists and } D^\alpha u \in L^p(\Omega) \ \forall |\alpha| \le k\}.$$

The natural norm is

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}^p \right)^{1/2} & , p \in [1,\infty) \\ \sum_{|\alpha|\le k} \underbrace{\|D^\alpha u\|_{L^\infty}} & , p = \infty . \end{cases}$$

essential supremum, i.e. $\|u\|_{L^\infty} = M$ if $u \le M$ a.e.

If Ω bdd, then $\|u\|_{H_0^1(\Omega)} = \int_\Omega |Du|^2 dx$ (you probably seen this somewhere).

(Hilbert spaces)

Remarks: (1) For $p=2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$, $k = 0,1,\dots$. So $H^0(\Omega) = L^2(\Omega)$.

(2) Sobolev functions are identified a.e.

(3) The choice of convention in measuring L^p norm of $D^\alpha u$ is not relevant in most cases, since all these are equivalent up to multiplicative constants.

An equivalent norm will be $\|u\|_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}$

(4) We say that $u_m \to u$ in $W^{k,p}(\Omega)$ if $\|u_m - u\|_{W^{k,p}(\Omega)} \to 0$ as $m \to \infty$.

We say that $u_m \to u$ in $W^{k,p}_{loc}(\Omega)$ if $\|u_m - u\|_{W^{k,p}(\tilde\Omega)} \to 0$ as $m \to \infty$ $\forall \tilde\Omega \subset\subset \Omega$.

Definition: For $\Omega \subset \mathbb{R}^n$ open, $1 \le p \le \infty$, $k \in \mathbb{N}_0$, the Sobolev space $W_0^{k,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$, i.e.

$$u \in W_0^{k,p}(\Omega) \iff \exists \{u_m\} \in C_c^\infty(\Omega) \text{ s.t. } u_m \to u \text{ in } W^{k,p}(\Omega).$$

~~(1) It might be that $W_0^{k,p}(\Omega)$ is just the same as $W^{k,p}(\Omega)$, but~~

(1) It might be that $W_0^{k,p}(\Omega)$ is just $W^{k,p}(\Omega)$ itself, but in general $W_0^{k,p}(\Omega) \subsetneq W^{k,p}(\Omega)$. Example: $u \equiv 1$ is $W^{1,p}(B)$ but not in $W_0^{1,p}(B)$ [unable to cut off u by changing the $W^{k,p}$ norm]

(2) For $1 \le p < \infty$, $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. [because $C_c^\infty(\cancel{\Omega}\ \mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$]

(3) We can think of ~~$W_0^{k,p}(\mathbb{R}^n)$~~ $W_0^{k,p}(\Omega)$ as functions in $W^{k,p}(\Omega)$ that are "zero on the boundary $\partial\Omega$". A-priori, this doesn't make sense at all since Sobolev or L^p functions are defined a.e. and typically $\partial\Omega$ has measure zero.

Exercises: For $n=1$, Ω open interval in $\mathbb{R}$, then ~~$u \in W^{1,p}($~~

$W^{1,1}$ = space of absolutely continuous functions on Ω [basically FTC for Lebesgue]

$W^{1,\infty}(\Omega)$ = space of Lipschitz function on Ω.

$W^{1,p}(\Omega)$ = space of absolutely continuous functions whose classical derivative $\in L^p(\Omega)$.

In general, however, Sobolev functions can be discontinuous and/or unbounded.

Example: (Spike functions). Consider $u: B \to \mathbb{R}$, $B \subset \mathbb{R}^n$, $\alpha > 0$, $u(x) = |x|^{-\alpha}$, $x \neq 0$.

$p \in [1,\infty)$. We show that u is weakly diff if $\alpha \in (0, n-1)$. Now, ~~when is $Du \in L^p(B)$??~~

$\boxed{1}$ $u \in L^p(\Omega)$??: $\|u\|_{L^p(\Omega)}^p = \int_B |x|^{-\alpha p}\,dx = C\int_0^1 r^{-\alpha p} r^{n-1}\,dr$

$$= C\int_0^1 r^{n-\alpha p-1}\,dr < \infty,$$

provided $n - \alpha p - 1 > -1$, $n - \alpha p > 0$, $n > \alpha p$, $\alpha < \frac{n}{p}$.

(i.e. $\alpha < n$ and p not too "large").

$\boxed{2}$ Since $D_i u = \frac{-\alpha x_i}{|x|^{\alpha+2}}$, $|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}}$, $x \neq 0$. A similar calculation shows that $(\alpha+1) < n$ and p not too large.

$Du \in L^p(B) \iff (\alpha+1)p < n$, or $\alpha < \frac{n}{p} - 1 = \frac{n-p}{p}$.

Hence, we conclude that $u \in W^{1,p}(B) \iff \alpha < \frac{n-p}{p}$. In particular, $u \notin W^{1,p}(B)\ \forall p \ge n$.

Cool thing: $\alpha < \frac{n-p}{p}$, $\alpha n p < n(n-p)$, $\alpha\left(\frac{np}{n-p}\right) < n$

$\therefore$ For spike functions u, $u \in W^{1,p}(B) \Rightarrow u \in L^{\frac{np}{n-p}}(B)$

We actually get more regularity, since $\frac{np}{n-p} > p$. [$L^{\frac{np}{n-p}}$ is "better" than L^p locally].

$\square$

Remarks: 1) Actually, Sobolev embedding thm will give us the result that the above is true for general Sobolev functions, i.e. $W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega)$ for $p \in [1, n)$. [C^1 boundary]

Even better for $W_0^{1,p}(\Omega)$, no need $\partial\Omega \in C^1$.

2) The same computation on spike functions for $W^{k,p}$ also shows that

$$u \in W^{k,p}(B) \Leftrightarrow (\alpha+k)p < n \text{ . or } \alpha < \frac{n}{p} - k = \frac{n-kp}{p}$$

This says that $u \in W^{k,p}(B) \Rightarrow u \in L^{\frac{np}{n-kp}}(B)$ [even more regular]

3) In some sense, if $p > n$, then we will "jump" over L^∞, into Hölder's space, more precisely, $C^{0,1-\frac{n}{p}}(\Omega)$.

4) For $p = n$, it seems like $W^{1,n} \hookrightarrow$ "$L^{\frac{n \cdot n}{n-n}} = L^\infty$". This is true for $n=1$, since $W^{1,1}(a,b)$ = space of absolutely continuous functions. Unfortunately, this fails for $n=2$.

Thm: (Properties of weak derivatives) Assume ~~u,~~ $u, v \in W^{k,p}(\Omega)$, $|\alpha| \le k$. Then

(Commute) (1) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$, $|\alpha| + |\beta| \le k$.

(2) $\forall \lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \le k$.

(Restriction) (3) ~~If $V \subset U$, V open, then $u \in W^{k,p}$~~ If $\tilde{\Omega} \subset \Omega$, Ω open, then $u \in W^{k,p}(\tilde{\Omega})$.

(4) If $\phi \in C_c^\infty(\Omega)$, then $\phi u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\phi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \phi \, D^{\alpha-\beta} u \quad \text{(Leibniz's formula)}$$

$\int u|_{\tilde{\Omega}} D^\alpha \phi$
$= \int u D^\alpha \phi$
$= (-1)^{|\alpha|} \int v \phi$
$= (-1)^{|\alpha|} \int v|_{\tilde{\Omega}} \phi|_{\tilde{\Omega}}$
$= (-1)^{|\alpha|} \int D^\alpha(u|_{\tilde{\Omega}}) \phi$
$\forall \phi \in C_c^\infty(\tilde{\Omega})$.

Thm: [Sobolev is Banach] For each $k=1,2,\dots$, $p\in[1,\infty]$, $W^{k,p}(\Omega)$ is a Banach space.

Proof: (1) First, check the norm. This follows from Minkowski inequality (duhh...).

(2) Let's show that $W^{k,p}(\Omega)$ is complete. Take a Cauchy sequence $\{u_m\}\in W^{k,p}(\Omega)$.

For each $|\alpha|\le k$, $\{D^\alpha u_m\}$ is a Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, $\exists\ u_\alpha\in L^p(\Omega)$ s.t. $D^\alpha u_m\to u_\alpha$ in $L^p(\Omega)$ $\forall|\alpha|\le k$.

In particular, $u_m\to u_{(0,0,\dots,0)}=:u$ in $L^p(\Omega)$.

Claim: This u is the $W^{k,p}$ limit, with $D^\alpha u=u_\alpha$, $|\alpha|\le k$.

For all $\phi\in C_c^\infty(\Omega)$,

Weak derivative of a limit = limit of a weak derivative.

$f_n\to f$, $D^\alpha f_n\to g$ $\Rightarrow$ $g=D^\alpha f$

$$\int_\Omega u D^\alpha\phi\,dx=\lim_{m\to\infty}\int_\Omega u_m D^\alpha\phi\,dx$$

(Use Hölder's to justify this). $C_c^\infty\subset L^q$

$$=\lim_{m\to\infty}(-1)^{|\alpha|}\int_\Omega D^\alpha u_m\phi\,dx$$

$$=(-1)^{|\alpha|}\int_\Omega u_\alpha\phi\,dx$$

$\blacksquare$

Since $D^\alpha u_m\to D^\alpha u$ in $L^p(\Omega)$ $\forall|\alpha|\le k$, we see that $u_m\to u$ in $W^{k,p}(\Omega)$.

$u\mapsto(u,D^\alpha u,\dots D^\alpha u)$

$\blacksquare$

Lemma: The spaces $W^{k,p}(\Omega)$ are

(a) Banach spaces for $p\in[1,\infty]$

(b) Separable for $p\in[1,\infty)$

(c) Reflexive for $1<p<\infty$.

The above actually shows that $W^{k,p}(\Omega)$ is a closed subspace of $L^p(\Omega_k)$ under this embedding. Done!!

Closed subspaces inherits completeness, reflexiveness and separableness.

Pf: Consider copies of Ω, i.e. $\Omega_k=\bigcup_{|\alpha|\le k}\Omega\times\{\alpha\}$, i.e. one copy for each multiindex.

Consider embedding of $W^{k,p}(\Omega)$ onto $L^p(\Omega_k)$

$u\mapsto u^{(k)}$, where $u^k(x,\alpha)=D^\alpha u$, $x\in\Omega$.

Note that $\|u\|_{W^{k,p}(\Omega)}=\|u^{(k)}\|_{L^p(\Omega_k)}$.